

Thermal transport through non-ideal Andreev quantum dots

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Abstract

We consider the scenario of thermal transport through two types of Andreev quantum dots which are coupled to two leads, belonging to the Class D and Class C symmetry classes. Using the random matrix description we derive the joint probability density function (j.p.d.f.) in term of Hypergeometric Function of Matrix Arguments when we consider one lead to be attached ideally and one lead non ideally. For the class C ensemble we derive a more explicit representation of the j.p.d.f. which results in a new type of random matrix model.

1 Introduction

Quantum dots form an important class of mesoscopic systems whose electric and thermal transport properties are being actively studied. Random matrix theory has had a tremendous success in describing these systems in the limit of low voltage and low temperature, when the classical motion of an electron is chaotic in the dot. In this regime transport through the dot boils down to characterizing how single (quasi-)particles are transmitted through a chaotic cavity. The Landauer-Büttiker approach does this through the study of the scattering matrix. If the classical dynamics is chaotic in the cavity then the scattering matrix would be well described by a random uniformly distributed unitary matrix [4], an ensemble which had already been studied by Dyson. Taking the distribution to be uniform over the scattering matrix gave rise to three ensembles, namely the circular unitary, orthogonal and symplectic ensemble or CUE, COE and CSE. Each of these is determined by the presence or absence of time reversal symmetry and spin rotation symmetry of the electron in the chaotic cavity. In [5] Altland and Zirnbauer showed the existence of four more types of symmetry classes which appear when particle-hole symmetry is present. When a quantum dot is put in contact with a superconductor an electron moving inside the cavity can be reflected as a hole. This process is called Andreev reflection [3] and in [5] it was shown that these hybrid normal metal-superconductor systems formed four new symmetry classes named class D, C, DIII and CI. When considering a quantum dot within these symmetry classes and assuming that the scattering matrix is uniformly distributed these give rise to the Circular Real Ensemble (CRE), Circular Quaternion Ensemble (CQE), Circular Real Time reversal symmetric Ensemble (T-CRE) and the Circular Quaternion Time reversal symmetric Ensemble (T-CQE), [14, 12]. Since Andreev reflections change the charge of the particle moving in the cavity, electric transport is no longer the same as thermal transport, which in these systems is the same as particle transport. Put differently, a particle scattered through the cavity will transport a definite amount of energy but not a definite amount of charge since it can come out as an electron or a hole.

The uniform distribution over all of these circular ensembles is the most “simple” scenario which, although it can be realized experimentally, need not be the case. It was shown in [6] that, if on average the scattering matrix was different from zero then the distribution over the scattering matrix is given by the Poisson Kernel, $P(S)$. This was shown for the CUE, COE and CSE cases. A generalization of the Poisson Kernel for the CRE, CQE, T-CRE and T-CQE was derived in [13]. When the distribution is given by the Poisson kernel, or its generalization, the system is said to be non ideal. When the leads are ideally coupled to the chaotic cavity the scattering matrix distribution is uniform.

Aside from being a more general description of the quantum dots, non ideal systems can be attractive for different reasons. For example, in [17] it was shown that semi-non-ideal quantum dots could be used to tune the amount of entanglement between two electrons scattering on the quantum dot.

Many transport observables, such as the conduction or the shot noise, can be written down in terms of the transmission or the reflection eigenvalues. The main obstacle when studying non ideal scenarios is that the joint probability density function (j.p.d.f.) for these eigenvalues is not available while it is available for the ideal case. Exceptions to this are the cases of the semi-non-ideal quantum dot with broken/preserved time reversal symmetry and spin rotation symmetry. These are non ideal versions of the CUE, solved in [15] and studied in [17] regarding entanglement and the COE and CSE analyzed in [16].

We will consider the problem of thermal transport through Andreev quantum dots where two leads are attached and we will consider only one lead to be non ideal (semi-non-ideal quantum dot). The symmetry classes we will analyze are the Class D and C. Class D systems correspond to those with broken time-reversal and spin-rotation symmetry while class C has only broken time-reversal

symmetry. This means we are looking at the non ideal version of the CRE and CQE, and refer to them as the Poisson Real and Quaternion Ensemble (PRE and PQE). In these cases the scattering matrices are orthogonal ($O(N)$) and symplectic ($Sp(N)$) respectively. The orthogonal scattering matrices can be further split into two parts. Matrices with determinant 1 and matrices with determinant -1 . The determinant is called the topological quantum number and when it is 1 (-1) we are in the topologically (non-)trivial phase. We will consider the case when the determinant is 1 and thus the scattering matrices form the group $SO(N)$.

Our strategy is analogous to the one used in previous work [16]. In section 2 we will review the Landauer-Büttiker approach and explain where the main hurdle lies to find the j.p.d.f.. In section 3 we will use the theory of symmetric function to derive how the j.p.d.f. can be expressed in terms of Hypergeometric Function of Matrix Arguments (HFMA). In B we review the main results from the theory of symmetric functions that we use in this derivation. In section 5 we will derive a representation of the HFMA which will be useful to derive a more compact representation to the j.p.d.f. for the quaternion ensemble.

2 Landauer-Büttiker approach

The system we consider is an Andreev quantum dots with a left lead with n channels and a right lead with m channels. We take $n \leq m$ and for the real ensemble we have included the spin and particle/hole quantum numbers. For the quaternion ensemble we do not include spin quantum number. The scattering matrix, S , is then a $(n+m) \times (n+m)$ matrix for the real ensemble and for the quaternion ensemble it is a $(n+m) \times (n+m)$ matrix with quaternion elements. Which means the scattering matrix is either an orthogonal matrix in $SO(N)$ or a symplectic matrix in $Sp(N)$. The transmission matrix, $\mathbf{t}_{n \times m}$, is a sub-block of the scattering matrix.

$$S = \begin{pmatrix} \mathbf{r}_{n \times n} & \mathbf{t}_{n \times m} \\ \mathbf{t}'_{m \times n} & \mathbf{r}'_{m \times m} \end{pmatrix}. \quad (2.1)$$

The Landauer-Büttiker approach characterizes transport through a quantum dot by the eigenvalues of the product of the transmission matrix with its hermitian conjugate. That is to say, the eigenvalues T_j of the matrix $\mathbf{t}\mathbf{t}^\dagger$ determine the thermal transport observables, such as the conductance G , through the following formula:

$$G = dG_0 \sum_{j=1}^n T_j, \quad (2.2)$$

with $G_0 = \frac{\pi^2 k_B^2 T_0}{6h}$ and d denotes the degeneracy of the transmission eigenvalues. Alternatively the reflection eigenvalues R_j , the eigenvalues of the matrix $\mathbf{r}\mathbf{r}^\dagger$, can be used. They are related to the transmission eigenvalues as $R_j = 1 - T_j$ and we will use these instead of the transmission eigenvalues. The Random Matrix Theory description of quantum dots start with a given distribution, $P(S)$, over the scattering matrix. Given this distribution the expectation value of an observable depending on the transmission eigenvalues, $F(R_j)$, is given by

$$\langle F(R_j) \rangle = \int d\mu(S) P(S) F(R_j), \quad (2.3)$$

where $d\mu(S)$ is the uniform measure or Haar measure over S . In order to characterize the statistics of observables depending on the reflection eigenvalues one needs to derive from $P(S)$ the joint probability density function (j.p.d.f.) of the reflection eigenvalues, $\mathcal{P}(R_j)$.

By using the polar decomposition of the scattering matrix it is parametrized as follows

$$S = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} r & t \\ -t' & r' \end{pmatrix} \begin{pmatrix} V_1^\dagger & 0 \\ 0 & V_2^\dagger \end{pmatrix} \quad (2.4)$$

$$= U \begin{pmatrix} r & t \\ -t' & r' \end{pmatrix} V^\dagger$$

where r, t, t' and r' are diagonal matrices. r has as diagonal values r_1, \dots, r_n , with $r_j \in [0, 1]$, and the reflection eigenvalues R_j are given by r_j^2 . r' the same values as r and an extra $m - n$ of 1's as diagonal values.

$$r = \text{diag} \{r_1, \dots, r_n\}$$

$$r' = \text{diag} \{r_1, \dots, r_n, 1 \dots 1\}$$

t on the other hand is rectangular ($n \times m$) with $n \leq m$ and has as diagonal elements t_1, \dots, t_n . While t' is the transpose of t , $t' = t^T$. In order for this parametrization to be unique we take $U_1, V_1 \in O(n)$, $V_2 \in O(m)$ and $U_2 \in O(m)/O(m-n)$ when $S \in SO(N)$. When $S \in Sp(N)$ we take $U_1 \in Sp(n)/Sp(1)^n$, $V_1 \in Sp(n)$, $V_2 \in Sp(m)$ and $U_2 \in Sp(m)/Sp(m-n)$. However, as shown in section A, the integrals over the coset spaces can be extended to the full group once the Jacobian is computed. Additionally, for the orthogonal matrices the determinant is equal to 1 and we need to insure that this condition is fulfilled in the parametrization. The determinant is given by

$$\det \left[\begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} r & t \\ -t' & r' \end{pmatrix} \begin{pmatrix} V_1^\dagger & 0 \\ 0 & V_2^\dagger \end{pmatrix} \right] = \det [U_1] \det [U_2] \det [V_1] \det [V_2].$$

Therefore we need to insure

$$\det [U_1] \det [U_2] \det [V_1] \det [V_2] = +1.$$

We set in our parametrization $\det [V_2]$ equal to $\det [V_1]$ and $\det [U_2]$ equal to $\det [U_1]$. Meaning the matrices U_2 and V_2 are constricted. Denoting by $J(r_j)$ the Jacobian of the transformation of Eq. (2.4) one gathers

$$\langle F(r_j) \rangle = \prod_{j=1}^n \int_0^1 dr_j F(r_j) J(r_j) \int d\mu(U) d\mu(V) P \left(U \begin{pmatrix} r & t \\ -t' & r' \end{pmatrix} V^\dagger \right) \quad (2.5)$$

For the quaternion ensemble $d\mu(U) d\mu(V)$ is a product of Haar measures over independent matrices

$$d\mu(U) d\mu(V) = d\mu(U_1) d\mu(U_2) d\mu(V_1) d\mu(V_2) \quad (2.6)$$

For the orthogonal ensemble $d\mu(U) d\mu(V)$ is a product of Haar measures over matrices whose determinants are related

$$d\mu(U) d\mu(V) = d\mu(U_1) d\mu(U_2) d\mu(V_1) d\mu(V_2) \delta(\det [V_1] - \det [V_2]) \delta(\det [U_1] - \det [U_2]) \quad (2.7)$$

with $U_j, V_j \in O(n)$ for the real ensemble and $U_j, V_j \in Sp(n)$ for the quaternion ensemble. We note also that for the quaternion ensemble the matrix is made of quaternion elements and so the diagonal quaternion matrix r of singular values has n blocks $r_j \mathbb{I}_{2 \times 2}$. Thus the singular values r_j are double

degenerate. The j.p.d.f., denoted by $\mathcal{P}(R_j)$, can almost be read of Eq. (2.5). Given that $r_j^2 = R_j$ we only need to make a change of variables in Eq. (2.5).

$$\begin{aligned}\langle F(R_j) \rangle &= \langle F(r_j^2) \rangle = \prod_{j=1}^n \int_0^1 dr_j F(r_j^2) J(r_j) \int d\mu(U) d\mu(V) P \left(U \begin{pmatrix} r & t \\ -t' & r' \end{pmatrix} V^\dagger \right) \\ &= \prod_{j=1}^n \int_0^1 dR_j F(R_j) J(\sqrt{R_j}) (2R_j)^{-\frac{1}{2}} \int d\mu(U) d\mu(V) P \left(U \begin{pmatrix} \sqrt{R} & \sqrt{T} \\ -\sqrt{T'} & \sqrt{R} \end{pmatrix} V^\dagger \right) \quad (2.8)\end{aligned}$$

The j.p.d.f. is then given by

$$\mathcal{P}(R_j) = J(\sqrt{R_j}) (2R_j)^{-\frac{1}{2}} \int d\mu(U) d\mu(V) P \left(U \begin{pmatrix} \sqrt{R} & \sqrt{T} \\ -\sqrt{T'} & \sqrt{R} \end{pmatrix} V^\dagger \right). \quad (2.9)$$

When both leads are attached to the quantum dot ideally random matrix theory models the ensemble of scattering through circular ensembles, meaning $P(S) = 1$. The ensemble generated by the orthogonal matrices is then called the Circular Real Ensemble (CRE) and the one generated by the symplectic matrices is called the Circular Quaternion Ensemble (CQE). The j.p.d.f. for this case was derived in [2], [12].

$$\mathcal{P}(R_j) \propto |\Delta(R_j)|^\beta (1 - R_j)^{\frac{\beta}{2}(m-n+1)-1} R_j^{\frac{n}{2}}$$

with the following values of β, η and d depending on the ensemble

Ensemble	β	η	d
CRE	1	-1	1
CQE	4	2	4

A more general situation is described when one allows for a non-ideal coupling between the leads and the dot. In this situation the distribution over the scattering matrix is a Poisson type kernel [13]

$$P(S) = \frac{1}{\mathcal{C}(\hat{\gamma})} |\det[1 - \hat{\gamma}S]|^{-N_\sigma}, \quad (2.10)$$

$$N_\sigma = N + \sigma$$

where $N = n + m$ for the real ensemble and $N = 2n + 2m$ for the quaternion ensemble. $\mathcal{C}(\hat{\gamma})$ is the normalization constant to be computed later on. These ensembles are no longer circular and we will refer to them as the Poisson Real Ensemble (PRE) and the Poisson Quaternion Ensemble (PQE). The parameter σ depends on the ensemble is given below.

PRE	$\sigma = -1$
PQE	$\sigma = 1$

(2.11)

The matrix $\hat{\gamma}$ encodes the coupling between the left/right lead and the dot.

$$\hat{\gamma} = \begin{pmatrix} \hat{\gamma}_L & 0 \\ 0 & \hat{\gamma}_R \end{pmatrix}$$

The left lead is taken to be non ideally coupled, $\hat{\gamma}_L \neq 0$, while the right one is arbitrary $\hat{\gamma}_R = 0$. We call this the semi-non-ideal scenario. Since we are studying the case where the right lead has coupling $\hat{\gamma}_R = 0$ the Poisson like kernel simplifies to

$$P(S) = \frac{1}{\mathcal{C}(\hat{\gamma})} |\det[1 - \hat{\gamma}_L \mathbf{r}]|^{-N_\sigma}$$

For the semi-non-ideal system we gather then from Eqs. (2.9), (2.6) and (2.7)

$$\mathcal{P}(R_j) = J(\sqrt{R_j}) (2R_j)^{-\frac{1}{2}} \int d\mu(U_1) d\mu(V_1) \left| \det \left[1 - \gamma_L U_1 r V_1^\dagger \right] \right|^{-(N+\sigma)}, \quad (2.12)$$

where the integrals are either over $O(n)$ or $Sp(n)$. The problem of finding the j.p.d.f. thus boils down to performing the integration over the orthogonal/symplectic group. Since we rely heavily on the theory of symmetric functions we have included an appendix where the most important features of the theory, for our present calculations, are explained.

3 The joint probability density function

3.1 Poisson Real Ensemble

For both ensembles the strategy is the same but we will perform them separately for the sake of clearness. The idea is to expand the Poisson kernel in terms of symmetric functions in order to perform the integrations over the group. Once this is done the result will turn out to be known as Hypergeometric Functions of Matrix Argument(s) (HFMA). In section 5 we will elaborate on different representations of these HFMA.

For the PRE we expand the inverse determinant using Eq. (B.59) in terms of the schur functions $S_\lambda(X)$. The integral to be performed is denoted by $I_{PRE}(\hat{\gamma}, R_j)$ and defined as follows:

$$\begin{aligned} I_{PRE}(\hat{\gamma}, R_j) &= \int_{O(n)} d\mu(U) d\mu(V) \det \left[1 - V^\dagger \hat{\gamma} U r \right]^{-(N+\sigma)} \\ &= \int_{O(n)} d\mu(U) d\mu(V) \sum_{\lambda} S_{\lambda}(\mathbb{I}_{N_\sigma}) S_{\lambda}(V^\dagger \hat{\gamma} U r) \\ &= \sum_{\lambda} S_{\lambda}(\mathbb{I}_{N_\sigma}) \int_{O(n)} d\mu(U) d\mu(V) S_{\lambda}(V^\dagger \gamma U r) \end{aligned}$$

with $\sigma = -1$ for this ensemble and \mathbb{I}_M denotes the identity matrix of dimension M . The integral over U (or V) is zero unless the partition is even [11]. This means the integers $\lambda = (\lambda_1, \lambda_2, \dots)$ defining the partition have to be even numbers. This is denoted by $\lambda = 2\lambda' = (2\lambda'_1, 2\lambda'_2, \dots)$. Thus the sum over partitions can be written as a sum over even partitions. For even partitions we have through Eq. (B.64)

$$\int_{O(n)} d\mu(U) S_{2\lambda}(AU) = \Omega_{\lambda}^{(2)}(A)$$

where $\Omega_{\lambda}^{(2)}(A)$ are called the spherical functions defined through their integral property Eq.(B.61). Thus using Eq. (B.61) yields

$$\begin{aligned} \int_{O(n)} d\mu(U) d\mu(V) S_{2\lambda}(V^\dagger \hat{\gamma} U r) &= \int_{O(n)} d\mu(V) \Omega_{\lambda}^{(2)}(r V^\dagger \hat{\gamma}) \\ &= \Omega_{\lambda}^{(2)}(\hat{\gamma}) \Omega_{\lambda}^{(2)}(r) \end{aligned}$$

The spherical functions can be expressed in terms of Jack Polynomials through Eq. (B.63). We find then

$$\int_{O(n)} dU dV S_{2\lambda}(V^\dagger \hat{\gamma} U r) = \frac{P_{\lambda}^{(2)}(\hat{\gamma}^2)}{P_{\lambda}^{(2)}(\mathbb{I}_n)} \frac{P_{\lambda}^{(2)}(r^2)}{P_{\lambda}^{(2)}(\mathbb{I}_n)}$$

Our integral is then

$$\begin{aligned} I_{PRE}(\hat{\gamma}, R_j) &= \sum_{\lambda} S_{2\lambda}(\mathbb{I}_{N_{\sigma}}) \frac{P_{\lambda}^{(2)}(\hat{\gamma}^2)}{P_{\lambda}^{(2)}(\mathbb{I}_n)} \frac{P_{\lambda}^{(2)}(R)}{P_{\lambda}^{(2)}(\mathbb{I}_n)} \\ &= \sum_{\lambda} \frac{e'_{\lambda}(2, N_{\sigma})}{d'_{\lambda}(2)} P_{\lambda}^{(2)}(\mathbb{I}_{N_{\sigma}}) \frac{P_{\lambda}^{(2)}(\hat{\gamma}^2)}{P_{\lambda}^{(2)}(\mathbb{I}_n)} \frac{P_{\lambda}^{(2)}(R)}{P_{\lambda}^{(2)}(\mathbb{I}_n)} \end{aligned}$$

where we have used Eq. (B.58) to obtain an expression for the Schur polynomial evaluated at even partitions. We can now express our result in terms of Pochhammer symbols using Eqs. (B.57) and (B.53).

$$I_{PRE}(\hat{\gamma}, R_j) = \sum_{\lambda} 2^{|\lambda|} \frac{\left[1 + \frac{(N_{\sigma}-1)}{2}\right]_{\lambda}^{(2)}}{d'_{\lambda}(2)} \frac{\left[\frac{N_{\sigma}}{2}\right]_{\lambda}^{(2)}}{\left[\frac{n}{2}\right]_{\lambda}^{(2)}} \frac{P_{\lambda}^{(2)}(\hat{\gamma}^2)}{P_{\lambda}^{(2)}(\mathbb{I}_n)} \frac{P_{\lambda}^{(2)}(R)}{P_{\lambda}^{(2)}(\mathbb{I}_n)} \quad (3.13)$$

We recognize that Eq. (3.13) is the definition of a Hypergeometric Function of two Matrix Arguments, HFMA₂, Eq. (B.66). There are three types HFMA₂, denoted by ${}_2\mathcal{F}_1^{(\alpha)}(a, b; c|X, Y)$ (with index $\alpha = 2, 1$ or $1/2$) and defined in terms of the symmetric functions as follows:

$${}_2\mathcal{F}_1^{(\alpha)}(a, b; c|X, Y) = \sum_{\lambda} \frac{\alpha^{|\lambda|}}{d'_{\lambda}(\alpha)} \frac{[a]_{\lambda}^{(\alpha)} [b]_{\lambda}^{(\alpha)}}{[c]_{\lambda}^{(\alpha)}} \frac{P_{\lambda}^{(\alpha)}(X) P_{\lambda}^{(\alpha)}(Y)}{P_{\lambda}^{(\alpha)}(\mathbb{I}_n)}$$

Thus we have setting $\sigma = -1$

$$I_{PRE}(\hat{\gamma}, R_j) = {}_2\mathcal{F}_1^{(2)}\left(\frac{N}{2}, \frac{N-1}{2}; \frac{n}{2} \middle| \hat{\gamma}, R\right) \quad (3.14)$$

Very little is actually known about determinantal/pfaffian representations of HFMA₂. If the coupling of the left lead to the quantum dot is independent of the mode then we are in the case where $\hat{\gamma} = \gamma \mathbb{I}$. The result reduces then to a HFMA₁, Eq. (B.67)

$$I_{PRE}(\gamma \mathbb{I}_n, R_j) = {}_2F_1^{(2)}\left(\frac{N}{2}, \frac{N-1}{2}; \frac{n}{2} \middle| \gamma^2 R\right) \quad (3.15)$$

Before turning to the question of representations of the HFMA₁ we analyze the PQE case in the same manner. The results will be HFMA_{1,2} with the index $\alpha = 1/2$.

3.2 Poisson Quaternion Ensemble

For quaternion ensemble we need to perform the following integration

$$I_{PQE}(\hat{\gamma}, R_j) = \int_{Sp(n)} d\mu(U) d\mu(V) |\det[1 - \hat{\gamma} U r V]|^{-N_{\sigma}} \quad (3.16)$$

Using the fact that the unitary matrices are symplectic we have $U^{\dagger} = U^R = -Z U^T Z$ with $Z = \mathbb{I}_n \otimes i\tau_y$ and τ_y the Pauli matrix, we gather that the determinant is real even though the matrix is complex.

$$\begin{aligned} \det[1 - \hat{\gamma} U r V]^* &= \det[1 - \hat{\gamma} U^* r V^*] \\ &= \det[1 - \hat{\gamma} Z U Z r Z V Z] \\ &= \det[1 - \hat{\gamma} U r V] \end{aligned}$$

In the last step we have used the fact that r has a double degeneracy, $r = \text{diag}\{r_1, \dots, r_n\} \otimes \mathbb{I}_{2 \times 2}$. Thus for the PQE we have

$$\begin{aligned} I_{PQE}(\hat{\gamma}, R_j) &= \int_{Sp(n)} d\mu(U) d\mu(V) \det[1 - \hat{\gamma} U r V]^{-N_\sigma} \\ &= \sum_{\lambda} \frac{[N_\sigma]_{\lambda}^{(1)}}{h_{\lambda}(1)} \int_{Sp(n)} d\mu(U) d\mu(V) S_{\lambda}(\hat{\gamma} U r V) \end{aligned}$$

Similarly to the PRE the integral will be zero for partitions which do not have a specific form, namely the form $\lambda = \lambda' \cup \lambda'$ for any λ' . The partition $\lambda' \cup \lambda'$ is defined by having each integer twice. That is $\lambda' \cup \lambda' = (\lambda'_1, \lambda'_1, \lambda'_2, \lambda'_2, \dots)$. We have then

$$I_{PQE}(\hat{\gamma}, R_j) = \sum_{\lambda} \frac{[N_\sigma]_{\lambda \cup \lambda}^{(1)}}{h_{\lambda \cup \lambda}(1)} \int_{Sp(n)} d\mu(U) d\mu(V) S_{\lambda \cup \lambda}(\hat{\gamma} U r V)$$

For partitions which do have this form we can perform the integrations using Eqs. (B.65), (B.62) and (B.63)

$$\begin{aligned} \int_{Sp(n)} d\mu(U) d\mu(V) S_{\lambda \cup \lambda}(\hat{\gamma} U r V) &= \int_{Sp(n)} d\mu(V) \Omega_{\lambda}^{(1/2)}(r V \hat{\gamma}) \\ &= \Omega_{\lambda}^{(1/2)}(\hat{\gamma}) \Omega_{\lambda}^{(1/2)}(r) \\ &= \frac{P_{\lambda}^{(1/2)}(\hat{\gamma}^2)}{P_{\lambda}^{(1/2)}(\mathbb{I}_n)} \frac{P_{\lambda}^{(1/2)}(R)}{P_{\lambda}^{(1/2)}(\mathbb{I}_n)} \end{aligned}$$

This leads to the following expression :

$$I_{PQE}(\hat{\gamma}, R_j) = \sum_{\lambda} \frac{e'_{\lambda}(\frac{1}{2}, \frac{N_\sigma}{2}) b_{\lambda}(\frac{1}{2}, \frac{N_\sigma}{2})}{d'_{\lambda}(\frac{1}{2}) h_{\lambda}(\frac{1}{2})} \frac{P_{\lambda}^{(\frac{1}{2})}(\hat{\gamma}^2)}{P_{\lambda}^{(\frac{1}{2})}(\mathbb{I}_n)} \frac{P_{\lambda}^{(\frac{1}{2})}(R)}{P_{\lambda}^{(\frac{1}{2})}(\mathbb{I}_n)}$$

where we have used Eq. (B.56). Using Eqs. (B.53) and (B.57) we have

$$I_{PQE}(\hat{\gamma}, R_j) = \sum_{\lambda} \left(\frac{1}{2}\right)^{|\lambda|} \frac{[N_\sigma - 1]_{\lambda}^{(1/2)} [N_\sigma]_{\lambda}^{(1/2)}}{d'_{\lambda}(\frac{1}{2}) [2n]_{\lambda}^{(1/2)}} \frac{P_{\lambda}^{(\frac{1}{2})}(\hat{\gamma}^2) P_{\lambda}^{(\frac{1}{2})}(R)}{P_{\lambda}^{(\frac{1}{2})}(\mathbb{I}_n)}$$

we identify this solution with the HFMA₂ the index with $\alpha = 1/2$. Setting $\sigma = 1$ we gather

$$\begin{aligned} I_{PQE}(\hat{\gamma}, R_j) &= {}_2\mathcal{F}_1^{(\frac{1}{2})}\left(N, N+1; 2n \middle| \hat{\gamma}^2, R\right) \\ &= {}_2\mathcal{F}_1^{(\frac{1}{2})}\left(2(n+m), 2(n+m)+1; 2n \middle| \hat{\gamma}^2, R\right) \end{aligned} \quad (3.17)$$

where we have set $\sigma = 1$ and the case of $\hat{\gamma}$ proportional to the identity yields

$$\begin{aligned} I_{PQE}(\gamma \mathbb{I}, R_j) &= {}_2F_1^{(\frac{1}{2})}\left(N, N+1; 2n \middle| \gamma^2 R\right) \\ &= {}_2F_1^{(\frac{1}{2})}\left(2(n+m), 2(n+m)+1; 2n \middle| \gamma^2 R\right) \end{aligned} \quad (3.18)$$

For the two ensembles we have then the following j.p.d.f. when $\hat{\gamma}$ is arbitrary

$$\mathcal{P}_\alpha(R_j) = \frac{1}{\mathcal{C}(\hat{\gamma})} \prod_{j=1}^n (R_j - 1)^{\frac{1}{\alpha}(m-n+1)-1} R_j^{\frac{n}{2}} |\Delta(R_j)|^{\frac{2}{\alpha}} {}_2F_1^{(\alpha)} \left(\frac{m+n}{\alpha}, \frac{m+n}{\alpha} + \frac{\eta}{2}; \frac{n}{\alpha} \middle| \hat{\gamma}^2, R \right), \quad (3.19)$$

and when $\hat{\gamma} = \gamma \mathbb{I}$ it simplifies to

$$\mathcal{P}_\alpha(R_j) = \frac{1}{\mathcal{C}(\hat{\gamma})} \prod_{j=1}^n (R_j - 1)^{\frac{1}{\alpha}(m-n+1)-1} R_j^{\frac{n}{2}} |\Delta(R_j)|^{\frac{2}{\alpha}} {}_2F_1^{(\alpha)} \left(\frac{m+n}{\alpha}, \frac{m+n}{\alpha} + \frac{\eta}{2}; \frac{n}{\alpha} \middle| \gamma^2 R \right) \quad (3.20)$$

where we have added the index α to the j.p.d.f. of Eq. (2.12) that specifies the ensemble. $\alpha = 2$ for the real ensemble and $\frac{1}{2}$ for the quaternion one. \mathcal{C} denotes the normalization constant which we compute now.

4 Normalization

From Eq. (3.19) we gather the normalization constant is given by the following integral

$$\begin{aligned} \mathcal{C}(\hat{\gamma}) &= \prod_{j=1}^n \int_0^1 dR_j (R_j - 1)^{\frac{1}{\alpha}(m-n+1)-1} R_j^{\frac{n}{2}} |\Delta(R_j)|^{\frac{2}{\alpha}} {}_2F_1^{(\alpha)} \left(\frac{m+n}{\alpha}, \frac{m+n}{\alpha} + \frac{\eta}{2}; \frac{n}{\alpha} \middle| \hat{\gamma}^2; R \right) \\ &= \sum_{\lambda} \frac{\alpha^{|\lambda|}}{d'_{\lambda}(\alpha)} \frac{\left[\frac{m+n}{\alpha} \right]_{\lambda}^{(\alpha)} \left[\frac{m+n}{\alpha} + \frac{\eta}{2} \right]_{\lambda}^{(\alpha)}}{\left[\frac{n}{\alpha} \right]_{\lambda}^{(\alpha)}} \frac{P_{\lambda}^{(\alpha)}(\hat{\gamma}^2)}{P_{\lambda}^{(\alpha)}(\mathbb{I}_n)} \prod_{j=1}^n \int_0^1 dR_j (R_j - 1)^{\frac{1}{\alpha}(m-n+1)-1} R_j^{\frac{n}{2}} |\Delta(R_j)|^{\frac{2}{\alpha}} P_{\lambda}^{(\alpha)}(R) \end{aligned}$$

Given the Selberg integral over Jack polynomials

$$\begin{aligned} \prod_{j=1}^n \int_0^1 dR_j (1 - R_j)^y R_j^x |\Delta(R_j)|^{\frac{2}{\alpha}} P_{\lambda}^{(\alpha)}(R) &= P_{\lambda}^{(\alpha)}(\mathbb{I}_n) \frac{\left[x + 1 + \frac{n-1}{\alpha} \right]_{\lambda}^{\alpha}}{\left[x + y + 2 + \frac{2}{\alpha}(n-1) \right]_{\lambda}^{\alpha}} S_n(x, y, \alpha) \\ S_n(x, y, \alpha) &= \prod_{j=1}^n \int_0^1 dR_j (1 - R_j)^y R_j^x |\Delta(R_j)|^{\frac{2}{\alpha}} \\ &= \prod_{j=0}^{n-1} \frac{\Gamma\left(x + 1 + \frac{j}{\alpha}\right) \Gamma\left(y + 1 + \frac{j}{\alpha}\right) \Gamma\left(1 + \frac{j}{\alpha}\right)}{\Gamma\left(x + y + 2 + \frac{n+j-1}{\alpha}\right) \Gamma\left(1 + \frac{j}{\alpha}\right)} \quad (4.21) \end{aligned}$$

we have

$$\begin{aligned} \prod_{j=1}^n \int_0^1 dR_j (R_j - 1)^{\frac{1}{\alpha}(m-n+1)-1} R_j^{\frac{n}{2}} |\Delta(R_j)|^{\frac{2}{\alpha}} P_{\lambda}^{(\alpha)}(R) &= C_n P_{\lambda}^{(\alpha)}(\mathbb{I}_n) \frac{\left[\frac{\eta}{2} + 1 - \frac{1}{\alpha} + \frac{n}{\alpha} \right]_{\lambda}^{\alpha}}{\left[\frac{\eta}{2} + 1 - \frac{1}{\alpha} + \frac{1}{\alpha}(m+n) \right]_{\lambda}^{\alpha}} \\ C_n &= S_n \left(\frac{\eta}{2}, \frac{1}{\alpha}(m-n+1) - 1, \alpha \right) \quad (4.22) \end{aligned}$$

C_n is the normalization constant for the circular ensemble ($\hat{\gamma} = 0$). Given that $\frac{\eta}{2} + 1 - \frac{1}{\alpha} = 0$ for both ensembles we have

$$\int_0^1 dR_j (R_j - 1)^{\frac{1}{\alpha}(m-n+1)-1} R_j^{\frac{n}{2}} |\Delta(R_j)|^{\frac{2}{\alpha}} P_{\lambda}^{(\alpha)}(R) = C_n P_{\lambda}^{(\alpha)}(\mathbb{I}_n) \frac{\left[\frac{n}{\alpha} \right]_{\lambda}^{\alpha}}{\left[\frac{1}{\alpha}(m+n) \right]_{\lambda}^{\alpha}} \quad (4.23)$$

and so

$$\begin{aligned}
\mathcal{C}(\hat{\gamma}) &= C_n \sum_{\lambda} \frac{\alpha^{|\lambda|}}{d'_{\lambda}(\alpha)} \left[\frac{m+n}{\alpha} + \frac{\eta}{2} \right]_{\lambda}^{(\alpha)} P_{\lambda}^{(\alpha)}(\hat{\gamma}^2) \\
&= C_n \prod_{j=1}^n [1 - \gamma_j^2]^{-\left(\frac{m+n}{\alpha} + \frac{\eta}{2}\right)} \\
&= C_n \det [1 - \hat{\gamma}^2]^{-\left(\frac{m+n}{\alpha} + \frac{\eta}{2}\right)}
\end{aligned} \tag{4.24}$$

For the orthogonal ensembles this yields $\det [1 - \hat{\gamma}^2]^{-\left(\frac{N}{2} - \frac{1}{2}\right)}$ and for the quaternion we have $\det [1 - \hat{\gamma}^2 \otimes \mathbb{I}_2]^{-\left(\frac{N}{2} + \frac{1}{2}\right)}$

5 Representations of HFMA₁

In this section we will derive another integral representation for the HFMA₁. We will show that the following matrix integral

$$F_{a,b}^{p,\alpha}(X) = \frac{1}{Z_p(a,b)} \prod_{j=1}^p \int_0^{\infty} dy_j |\Delta(y_j)|^{\frac{2}{\alpha}} \frac{y_j^a}{(1+y_j)^b} \prod_{k,j}^{n,p} (1 + x_k y_j), \tag{5.25}$$

with $\alpha = 1/2, 1, 2$ and $Z_p(a,b)$ the normalization constant

$$\begin{aligned}
Z_p(a,b) &= \prod_{j=1}^p \int_0^{\infty} dy_j |\Delta(y_j)|^{\frac{2}{\alpha}} \frac{y_j^a}{(1+y_j)^b} \\
&= 2^{2(p-1)(1+\frac{1}{\alpha}(p-1))} \prod_{j=0}^{p-1} \frac{\Gamma\left(a+1+\frac{j}{\alpha}\right) \Gamma\left(b-a-1-\frac{2}{\alpha}(p-1)+\frac{j}{\alpha}\right) \Gamma\left(1+\frac{j}{\alpha}\right)}{\Gamma\left(b-\frac{1}{\alpha}(p-1)+\frac{j}{\alpha}\right) \Gamma\left(1+\frac{1}{\alpha}\right)}
\end{aligned} \tag{5.26}$$

is a HFMA₁. We first use the dual Cauchy identity

$$\prod_{j,k}^{n,p} (1 + x_j y_k) = \sum_{\lambda} P_{\lambda^t}^{(\frac{1}{\alpha})}(X) P_{\lambda}^{(\alpha)}(Y)$$

leading to

$$F_{a,b}^{p,\alpha}(X) = \frac{1}{Z_p(a,b)} \sum_{\lambda} P_{\lambda^t}^{(\frac{1}{\alpha})}(X) \prod_{j=1}^p \int_0^{\infty} dy_j |\Delta(y_j)|^{\frac{2}{\alpha}} \frac{y_j^a}{(1+y_j)^b} P_{\lambda}^{(\alpha)}(y_j) \tag{5.27}$$

The sum over λ is over partitions such that $l(\lambda) \leq p$ and $l(\lambda^t) \leq n$. The dual generalized Selberg integrals states the following identity holds

$$\prod_{j=1}^p \int_0^{\infty} dy_j |\Delta(y_j)|^{\frac{2}{\alpha}} \frac{y_j^a}{(1+y_j)^b} P_{\lambda}^{(\alpha)}(y_j) = Z_p(a,b) P_{\lambda}^{(\alpha)}(\mathbb{I}_p) \frac{\left[a+1+\frac{p-1}{\alpha}\right]_{\lambda}^{(\alpha)}}{(-1)^{|\lambda|} \left[a+2+2\frac{p-1}{\alpha}-b\right]_{\lambda}^{(\alpha)}}$$

provided $l(\lambda^t) < b-a-1-2\frac{p-1}{\alpha}$. Since the sum is over partitions such that $l(\lambda^t) \leq n$, the condition is fulfilled for all partitions if $n < b-a-1-2\frac{p-1}{\alpha}$. Let us assume this last inequality holds. Using

the dual generalized Selberg integral in Eq. (5.27) we have

$$F_{a,b}^{p,\alpha}(X) = \sum_{\lambda} \frac{\left[a + 1 + \frac{p-1}{\alpha} \right]_{\lambda}^{(\alpha)}}{(-1)^{|\lambda|} \left[a + 2 + 2\frac{p-1}{\alpha} - b \right]_{\lambda}^{(\alpha)}} P_{\lambda^t}^{(\frac{1}{\alpha})}(X) P_{\lambda}^{(\alpha)}(\mathbb{I}_p) \quad (5.28)$$

We would like to rewrite this expression solely in terms of λ^t and $\frac{1}{\alpha}$ so as to compare it with the definition of HFMA₁. For the Jack polynomial evaluated at identity we have

$$P_{\lambda}^{(\alpha)}(\mathbb{I}_p) = \frac{\alpha^{|\lambda|} \left[\frac{p}{\alpha} \right]_{\lambda}^{(\alpha)}}{h_{\lambda}(\alpha)} \quad (5.29)$$

and using the following relationship between Pochhammer symbols of different index α

$$[s]_{\lambda}^{(\alpha)} = \frac{(-1)^{|\lambda|}}{\alpha^{|\lambda|}} [-\alpha s]_{\lambda^t}^{(\frac{1}{\alpha})} \quad (5.30)$$

we can express the Jack polynomial evaluated at the identity $P_{\lambda}^{(\alpha)}(\mathbb{I}_p)$ as

$$P_{\lambda}^{(\alpha)}(\mathbb{I}_p) = \frac{(-1)^{|\lambda|} [-p]_{\lambda^t}^{(\frac{1}{\alpha})}}{h_{\lambda}(\alpha)} \quad (5.31)$$

Using Eq. (5.30) we can also rewrite the ratios of Pochhammer symbols in the sum Eq.(5.28) as

$$\frac{\left[a + 1 + \frac{p-1}{\alpha} \right]_{\lambda}^{(\alpha)}}{\left[a + 2 + 2\frac{p-1}{\alpha} - b \right]_{\lambda}^{(\alpha)}} = \frac{\left[-\alpha \left(a + 1 + \frac{p-1}{\alpha} \right) \right]_{\lambda^t}^{(\frac{1}{\alpha})}}{\left[-\alpha \left(a + 2 + 2\frac{p-1}{\alpha} - b \right) \right]_{\lambda^t}^{(\frac{1}{\alpha})}} \quad (5.32)$$

In addition we have the following relations

$$h_{\lambda}(\alpha) = \alpha^{|\lambda|} d'_{\lambda^t} \left(\frac{1}{\alpha} \right) \quad (5.33)$$

Combining Eqs. (5.31), (5.32) and (5.33) in Eq. (5.28) we gather

$$F_{a,b}^{p,\alpha}(X) = \sum_{\lambda; l(\lambda) \leq p; l(\lambda^t) \leq n} \frac{[-p]_{\lambda^t}^{(\frac{1}{\alpha})}}{\alpha^{|\lambda|} d'_{\lambda^t} \left(\frac{1}{\alpha} \right)} \frac{[-\alpha(a+1) + 1 - p]_{\lambda^t}^{(\frac{1}{\alpha})}}{[-\alpha(a+2) + 2(1-p) + \alpha b]_{\lambda^t}^{(\frac{1}{\alpha})}} P_{\lambda^t}^{(\frac{1}{\alpha})}(X) \quad (5.34)$$

Since there is a one to one correspondence between partitions and their conjugates, summing over all partitions is the same as summing over all conjugate partitions. We make the change in notation $\lambda^t \rightarrow \lambda$ and denote $\alpha' = \frac{1}{\alpha}$ leading to

$$F_{a,b}^{p,\alpha}(X) = \sum_{\lambda; l(\lambda^t) \leq p; l(\lambda) \leq n} \frac{(\alpha')^{|\lambda|} [-p]_{\lambda}^{(\alpha')} \left[-\frac{1}{\alpha'}(a+1) + 1 - p \right]_{\lambda}^{(\alpha')}}{d'_{\lambda}(\alpha') \left[-\frac{1}{\alpha'}(a+2) + 2(1-p) + \frac{b}{\alpha'} \right]_{\lambda}^{(\alpha')}} P_{\lambda}^{(\alpha')}(X) \quad (5.35)$$

The Pochhammer symbol $[-p]_{\lambda}^{(\alpha')}$ is zero if $\lambda_1 > p$. Since $\lambda_1 = l(\lambda^t)$ the restriction $l(\lambda^t) \leq p$ is automatically satisfied in the sum. Thus we have

$$F_{a,b}^{p,\alpha}(X) = \sum_{\lambda; l(\lambda) \leq n} \frac{(\alpha')^{|\lambda|} [-p]_{\lambda}^{(\alpha')} [-q]_{\lambda}^{(\alpha')}}{d'_{\lambda}(\alpha') [c]_{\lambda}^{(\alpha')}} P_{\lambda}^{(\alpha')}(X) \quad (5.36)$$

with

$$-q = -\frac{1}{\alpha'}(a+1) + 1 - p \quad (5.37)$$

$$c = -\frac{1}{\alpha'}(a+2) + 2(1-p) + \frac{b}{\alpha'} \quad (5.38)$$

The sum in Eq. (5.36) is known to be a HFMA_1 , Eq. (B.67).

$$F_{a,b}^{p,\alpha}(X) = {}_2F_1^{(\alpha')}(-p, -q; c|X) \quad (5.39)$$

This identity holds subjected to the condition which came from the use of the dual Selberg integral.

$$n < b - a - 1 - 2\alpha'(p-1)$$

Thus for the HFMA_1 ${}_2F_1^{(\alpha')}(-p, -q; c|X)$ the condition translates into (with $c = \frac{1}{\alpha'}(b-a-2) - 2(p-1)$)

$$\frac{n-1}{\alpha'} < c \quad (5.40)$$

If the condition is met the HFMA_1 has the following integral representation

$$\begin{aligned} {}_2F_1^{(\alpha')}(-p, -q; c|X) &= \frac{1}{Z_p} \prod_{j=1}^p \int_0^\infty dy_j |\Delta(y_j)|^{\frac{2}{\alpha}} \frac{y_j^{\alpha'(q-p+1)-1}}{(1+y_j)^{\alpha'(c+p+q-1)+1}} \prod_{k,j}^{n,p} (1+x_k y_j) \\ Z_p &= \prod_{j=1}^p \int_0^\infty dy_j |\Delta(y_j)|^{\frac{2}{\alpha}} \frac{y_j^{\alpha'(q-p+1)-1}}{(1+y_j)^{\alpha'(c+p+q-1)+1}} \\ &= 2^{2(p-1)(1+\alpha'(p-1))} \prod_{j=0}^{p-1} \frac{\Gamma(\alpha'(q-j)) \Gamma(\alpha'(c+j)+1) \Gamma(1+j\alpha')}{\Gamma(\alpha'(c+q+j)+1) \Gamma(1+\alpha')} \end{aligned} \quad (5.41)$$

Performing the change of variables $y_j = \frac{1+\lambda_j}{1-\lambda_j}$ we have

$$\begin{aligned} &{}_2F_1^{(\alpha')}(-p, -q; c|X) \\ &= \frac{1}{\tilde{Z}_p} \prod_k^n (1-x_k)^p \prod_{j=1}^p \int_{-1}^1 d\lambda_j |\Delta(\lambda_j)|^{\frac{2}{\alpha}} (1+\lambda_j)^{\alpha'(q-p+1)-1} (1-\lambda_j)^{\alpha'c-n} \prod_{k,j}^{n,p} \left(\frac{1+x_k}{1-x_k} - \lambda_j \right) \quad (5.42) \\ \tilde{Z}_p &= \frac{Z_p}{2^{(p-1)(1+p\alpha')-\alpha'(c+p+q-1)}} \end{aligned}$$

and if we set $z_k = \frac{1+x_k}{1-x_k}$ then the integral is the average of a product of characteristic polynomials of a Jacobi Ensemble, when $\frac{2}{\alpha} = 1, 2$ and the average of a product of square roots of characteristic polynomials when $\frac{2}{\alpha} = 4$.

6 Representation of the j.p.d.f.

The representation of HFMA_1 derived above applies only for negative integer values of the first two arguments $-p, -q$ while in Eq. (3.20) the first two arguments of the HFMA_1 has clearly positive

values. To obtain a representation of the j.p.d.f. we use the well known Kummer's relations for the HFMA₁,

$${}_2F_1^{(\alpha)}(a, b; c | X) = \frac{{}_2F_1^{(\alpha)}(c - a, c - b; c | X)}{\det[1 - X]^{a+b-c}} \quad (6.43)$$

We first look at the PQE, $\alpha = \frac{1}{2}$. For the PQE we have from Eq. (3.18)

$$I_{PQE}(\gamma \mathbb{I}, R) = \frac{{}_2F_1^{(\frac{1}{2})}(-2m, -2m - 1; 2n | \gamma^2 R)}{\det[1 - \gamma^2 R]^{4m+2n+1}}$$

In Eq. (5.42) we set $p = \frac{m}{\alpha}$ and $q = \frac{m}{\alpha} + \frac{n}{2}$, and have then $p = 2m$, $q = 2m + 1$. For these values of p , q and c the weight in (5.42) simplifies to $(1 + \lambda_j)^{\alpha(q-p+1)-1} (1 - \lambda_j)^{\alpha c - n} = 1$. The HFMA₁ appearing can be expressed as a pfaffian over a Vandermonde determinant through Eqs.(C.71),(C.74) depending on whether n is even or odd. We assume n is even and using the result of Eq. (C.74) we gather

$$\begin{aligned} {}_2F_1^{(\frac{1}{2})}(-2m, -2m - 1; 2n | \gamma^2 R) &= \frac{1}{\tilde{Z}_{2m}} \prod_{k=1}^n (1 - \gamma^2 R_k)^{2m} \prod_{j=1}^{2m} \int_{-1}^1 d\lambda_j |\Delta(\lambda_j)| \prod_{k,j}^{n, 2m} \left(\frac{1 + \gamma^2 R_k}{1 - \gamma^2 R_k} - \lambda_j \right) \\ &= \frac{1}{\tilde{Z}_{2m}} \prod_{k=1}^n (1 - \gamma^2 R_k)^{2m} \frac{1}{\Delta\left(\frac{1 + \gamma^2 R_k}{1 - \gamma^2 R_k}\right)} \text{Pf}_{j,k \leq n} [f_{jk}] \\ &= \frac{1}{\tilde{Z}_{2m} (2\gamma^2)^{\frac{n(n-1)}{2}}} \prod_{k=1}^n (1 - \gamma^2 R_k)^{2m+n-1} \frac{1}{\Delta(R_k)} \text{Pf}_{j,k \leq n} [f_{jk}] \end{aligned}$$

with

$$\tilde{Z}_{2m} = 2^{2m^2+2+n} \prod_{j=0}^{2m-1} \frac{\Gamma\left(\frac{1}{2}(2m+1-j)\right) \Gamma\left(\frac{1}{2}(2n+j)+1\right) \Gamma\left(1+j\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}(2n+2m+1+j)+1\right) \Gamma\left(1+\frac{1}{2}\right)}$$

and the entries f_{jk} are given in terms of the Jacobi skew orthogonal polynomials by Eqs. (C.75) with the weight $w(u) = 1$. The arguments v_k in Eqs. (C.75) are $\frac{1 + \gamma^2 R_k}{1 - \gamma^2 R_k}$. The j.p.d.f. is then for the PQE

$$\mathcal{P}_{\alpha=\frac{1}{2}}(R_j) = \frac{1}{C(\gamma)} \Delta(R_j)^3 \prod_{j=k}^n \frac{R_k^{\frac{n}{2}} (1 - R_k)^{2(m-n)+1}}{(1 - \gamma^2 R_k)^{2m+n+2}} \text{Pf}[f_{jk}] \quad (6.44)$$

$$C(\gamma) = C(\hat{\gamma}) \tilde{Z}_{2m} (2\gamma^2)^{n \frac{(n-1)}{2}}$$

Given the antisymmetry of the Vandermonde determinant and the pfaffian under exchange of two variables R_j and R_k we gather

$$\mathcal{P}_{\alpha=\frac{1}{2}}(R_j) = \frac{1}{C(\hat{\gamma})} \Delta(R_j)^3 \prod_{k=1}^n \frac{R_k^{\frac{n}{2}} (1 - R_k)^{2(m-n)+1}}{(1 - \gamma^2 R_k)^{2m+n+2}} \prod_{s=1}^{\frac{n}{2}} F\left(\frac{1 + \gamma^2 R_{2s-1}}{1 - \gamma^2 R_{2s-1}}, \frac{1 + \gamma^2 R_{2s}}{1 - \gamma^2 R_{2s}}\right) \quad (6.45)$$

where the function $F(u, v)$ is given by Eq. (C.76). For $\alpha = 2$ ($\beta = 1$, the PRE case) we have from Eq. (3.15)

$$I_{PRE}(\gamma \mathbb{I}, R) = \frac{{}_2F_1^{(2)}\left(-\frac{m}{2}, -\frac{m-1}{2}; \frac{n}{2} | \gamma^2 R\right)}{\det[1 - \gamma^2 R]^{\frac{2m+n-1}{2}}}$$

There are two possibilities, m even or odd. For m even we take $p = \frac{m}{2}$, $q = \frac{m-1}{2}$. For these values of p , q and c the weight in (5.42) simplifies to $(1 + \lambda_j)^{\alpha(q-p+1)-1} (1 - \lambda_j)^{\alpha c - n} = 1$.

$${}_2F_1^{(2)} \left(-\frac{m}{2}, -\frac{m-1}{2}; \frac{n}{2} \middle| \gamma^2 R \right) = \frac{1}{\tilde{Z}_{\frac{m}{2}}} \prod_{k=1}^n (1 - \gamma^2 R_k)^{\frac{m}{2}} \prod_{j=1}^{\frac{m}{2}} \int_{-1}^1 d\lambda_j |\Delta(\lambda_j)|^4 \prod_{k,j}^{n, \frac{m}{2}} \left(\frac{1 + \gamma^2 R_k}{1 - \gamma^2 R_k} - \lambda_j \right) \quad (6.46)$$

For m odd we take $p = \frac{m-1}{2}$, $q = \frac{m}{2}$. For these values of p , q and c the weight in (5.42) simplifies to $(1 + \lambda_j)^{\alpha(q-p+1)-1} (1 - \lambda_j)^{\alpha c - n} = (1 + \lambda_j)^2$.

$$\begin{aligned} & {}_2F_1^{(2)} \left(-\frac{m}{2}, -\frac{m-1}{2}; \frac{n}{2} \middle| \gamma^2 R \right) \\ &= \frac{1}{\tilde{Z}_{\frac{m-1}{2}}} \prod_{k=1}^n (1 - \gamma^2 R_k)^{\frac{m-1}{2}} \prod_{j=1}^{\frac{m-1}{2}} \int_{-1}^1 d\lambda_j |\Delta(\lambda_j)|^4 (1 + \lambda_k)^2 \prod_{k,j}^{n, \frac{m-1}{2}} \left(\frac{1 + \gamma^2 R_k}{1 - \gamma^2 R_k} - \lambda_j \right) \end{aligned} \quad (6.47)$$

The products appearing in the average here are not characteristic polynomials but rather square roots of characteristic polynomials and this is why we can not follow the same type of calculation as for the PQE.

7 Conclusion

We have shown that when considering thermal transport through a semi-non-ideal Andreev quantum dot the j.p.d.f. is related to Hypergeometric Functions of Matrix Argument quite analogously to the case of electric transport studied in [15] and [16]. In addition we have derived for the quaternion ensemble a different representation of j.p.d.f. and found a new type of random matrix model. These results can be used as a starting point for further analyzing thermal transport through such quantum dots.

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A Polar Decomposition

In this section we will discuss some details about the polar decomposition and the uniqueness of the decomposition. The scattering matrix can belong to $SO(N)$ or $Sp(N)$. The polar decomposition for the scattering matrix states that it can be decomposed as follows.

$$\begin{aligned} S &= \begin{pmatrix} \mathbf{r}_{n \times n} & \mathbf{t}_{n \times m} \\ \mathbf{t}'_{m \times n} & \mathbf{r}'_{m \times m} \end{pmatrix} \\ &= \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} r & t \\ -t' & r' \end{pmatrix} \begin{pmatrix} V_1^\dagger & 0 \\ 0 & V_2^\dagger \end{pmatrix} \\ &= U \begin{pmatrix} r & t \\ -t' & r' \end{pmatrix} V^\dagger \end{aligned}$$

where r, t, t', r' are diagonal matrices and we have taken $n \leq m$. The matrices $U_1, V_1 \in O(n)$ and $U_2, V_2 \in O(m)$ when $S \in SO(N)$ while the matrices $U_1, V_1 \in Sp(n)$, $U_2, V_2 \in Sp(m)$ when $S \in Sp(N)$. Given the unitarity condition on S we have the following relations among the diagonal elements of r, t, t', r'

$$\begin{aligned} r_j^2 + t_j^2 &= 1 \\ t'_j &= t_j \\ \text{for } j \leq n \quad r'_j &= r_j \\ \text{for } j > n \quad r'_j &= 1 \end{aligned} \tag{A.48}$$

However this decomposition is not unique. The matrix r' has all diagonal elements equal to 1 when $j > n$ which means it is invariant under a unitary transformation in this sector. The rectangular matrices t and t' is filled with 0's in this sector. Thus we can restrict U_2 to the coset space $O(m)/O(m-n)$ when the scattering matrix is in $SO(N)$ and to the coset space $Sp(m)/Sp(m-n)$ when the scattering matrix is in $Sp(N)$. The number of degrees of freedom of $SO(N)$ is given by $\frac{N(N-1)}{2}$. In our parametrization we have $\frac{n(n-1)}{2}$ degrees of freedom for U_1, V_1 , $\frac{m(m-1)}{2}$ degrees of freedom for V_2 and $\frac{m(m-1)}{2} - \left(\frac{(m-n)(m-n-1)}{2}\right)$ degrees of freedom for U_2 . Adding to these the n degrees of freedom from the r_j variables we have in total $\frac{n^2+m^2-n-m}{2} + mn$ which accounts for all the number of degrees of freedom of $SO(N)$, $\frac{N(N-1)}{2}$.

A similar situation presents itself for the decomposition of $Sp(N)$. The number of degrees of freedom for $Sp(N)$ is $N(2N+1)$. To make the parametrization unique we take $U_1 \in Sp(n)$, $V_1 \in Sp(n)/Sp(1)^n$, $U_2 \in Sp(m)/Sp(m-n)$ and $V_2 \in Sp(m)$. Summing up the degrees of freedom we have $2n^2 + 2m^2 + 4mn + m + n$ which correspond to the $N(2N+1)$ degrees of freedom of $Sp(N)$. A unique parametrization is necessary to compute the Jacobian. However the scattering matrix is invariant under the subgroup $O(m-n)$ for the case of $SO(N)$ and invariant under $Sp(m-n)$ and $Sp(1)^n$ in the case of $Sp(N)$. This means that we can extend the integration over the coset space to the group, the difference being a proportionality constant. More precisely we have for every matrix U of the form

$$\begin{aligned} U &= \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & U' \end{pmatrix} \\ U' &\in O(m-n), \end{aligned}$$

we have

$$F \left(\begin{pmatrix} U_1 r V_1 & U_1 t V_2 \\ -U_2 t' V_1 & U_2 r' V_2 \end{pmatrix} \right) = F \left(\begin{pmatrix} U_1 r V_1 & U_1 t V_2 \\ -U_2 U' t' V_1 & U_2 U' r' V_2 \end{pmatrix} \right).$$

Therefore the identity holds when integrating over U'

$$F \left(\begin{pmatrix} U_1 r V_1 & U_1 t V_2 \\ -U_2 t' V_1 & U_2 r' V_2 \end{pmatrix} \right) = \int_{O(m-n)} d\mu(U') F \left(\begin{pmatrix} U_1 r V_1 & U_1 t V_2 \\ -U_2 U' t' V_1 & U_2 U' r' V_2 \end{pmatrix} \right).$$

We have then

$$\begin{aligned} & \int_{SO(N)} d\mu(S) F(S) \\ &= \int dr_j J(r_j) \int_{O(n)} d\mu(U_1) d\mu(V_1) d\mu(V_2) \int_{O(n)/O(m-n)} d\mu(U_2) F \left(\begin{pmatrix} U_1 r V_1 & U_1 t V_2 \\ -U_2 t' V_1 & U_2 r' V_2 \end{pmatrix} \right) \\ &= \int dr_j J(r_j) \int_{O(n)} d\mu(U_1) d\mu(V_1) d\mu(V_2) \int_{O(n)/O(m-n)} d\mu(U_2) \int_{O(m-n)} d\mu(U) F \left(\begin{pmatrix} U_1 r V_1 & U_1 t V_2 \\ -U_2 U' t' V_1 & U_2 U' r' V_2 \end{pmatrix} \right) \\ &= \int dr_j J(r_j) \int_{O(n)} d\mu(U_1) d\mu(V_1) d\mu(V_2) d\mu(U_2) F \left(\begin{pmatrix} U_1 r V_1 & U_1 t V_2 \\ -U_2 t' V_1 & U_2 r' V_2 \end{pmatrix} \right) \end{aligned} \quad (\text{A.49})$$

B Theory of symmetric functions

We use here the theory of symmetric functions to expand a given symmetric function of multiple variables $f(x_1, \dots, x_n)$ in terms of Jack polynomials and subsequently integrate using some known integration properties of these polynomials.

B.1 Preliminaries and notation

A set of non increasing integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ is called a partition of κ if

$$\begin{aligned} \lambda &= (\lambda_1, \lambda_2, \dots, \lambda_l) \\ \lambda_j &\geq \lambda_{j+1} \\ \sum_{j=1}^l \lambda_j &= \kappa. \end{aligned}$$

κ is called the weight of the partition and the length of the partition, $l(\lambda) = l$, is the number of integers λ_j . Often one also writes $(\lambda_1, \lambda_2, \dots, \lambda_l, 0, \dots, 0) = (\lambda_1, \lambda_2, \dots, \lambda_l)$. To each partition there is an associated diagram made of boxes. For a partition λ there are λ_1 boxes in the first row, represented by a diagram λ_2 boxes in the second row and so on. The length, $l(\lambda)$, then denotes the amount of rows. Each box is then denoted by two coordinates $s = (i, j)$. The arm of a box $a_\lambda(s)$ is equal to the number of boxes directly to the right of the box $s = (i, j)$. The leg of a box $l_\lambda(s)$ is equal to the number of boxes directly below of the box $s = (i, j)$. Similarly the co-arm and co-leg are defined as the boxes directly to the left and above the box $s = (i, j)$. The conjugate of a partition λ is a partition denoted by λ^T and defined as

$$\lambda_k^T = \# \{ \lambda_j \in \lambda : \lambda_j \geq k \}$$

The generalized Pochhammer symbol is given by

$$[u]_\lambda^{(\alpha)} = \prod_{j>1} \frac{\Gamma \left(u - \frac{j-1}{\alpha} + \lambda_j \right)}{\Gamma \left(u - \frac{j-1}{\alpha} \right)} \quad (\text{B.50})$$

We define the following coefficients

$$\begin{aligned}
d_\lambda(\alpha) &= \prod_{s \in \lambda} (\alpha a(s) + \alpha + l(s) + 1) \\
d'_\lambda(\alpha) &= \prod_{s \in \lambda} (\alpha a(s) + \alpha + l(s)) \\
e_\lambda(\alpha, n) &= \prod_{s \in \lambda} (\alpha a'(s) + \alpha + n - l'(s)) \\
e'_\lambda(\alpha, n) &= \prod_{s \in \lambda} (\alpha a'(s) + \alpha + n - l'(s) - 1) \\
h_\lambda(\alpha) &= \prod_{s \in \lambda} (\alpha a(s) + l(s) + 1) \\
b_\lambda(\alpha, n) &= \prod_{s \in \lambda} (\alpha a'(s) + n - l'(s))
\end{aligned}$$

We have then in terms of the Pochhammer symbol

$$b_\lambda(\alpha, n) = \alpha^{|\lambda|} \left[\frac{n}{\alpha} \right]_\lambda^{(\alpha)} \quad (\text{B.51})$$

$$e_\lambda(\alpha, n) = \alpha^{|\lambda|} \left[1 + \frac{n}{\alpha} \right]_\lambda^{(\alpha)} \quad (\text{B.52})$$

$$e'_\lambda(\alpha, n) = \alpha^{|\lambda|} \left[1 + \frac{n-1}{\alpha} \right]_\lambda^{(\alpha)} \quad (\text{B.53})$$

B.1.1 A combinatorial identity

We have the following identity for $\alpha = 1/2$

$$S_{\lambda \cup \lambda}(\mathbb{I}_{2n}) = \frac{e'_\lambda(1/2, n) b_\lambda(1/2, n)}{d'_\lambda(1/2) h_\lambda(1/2)}$$

The right hand side can be written down in terms of Pochhammer symbols and because we have on the left hand side a schur function of the identity we can express this side also in terms of the Pochhammer symbol.

$$\frac{[2n]_{\lambda \cup \lambda}^{(1)}}{h_{\lambda \cup \lambda}(1)} = \frac{(1/2)^{2|\lambda|} [2n-1]_\lambda^{(1/2)} [2n]_\lambda^{(1/2)}}{d'_\lambda(1/2) h_\lambda(1/2)} \quad (\text{B.54})$$

This identity holds for n integer and we will show now it holds for n real. We denote by μ_j the j^{th} integer of the partition $\lambda \cup \lambda$. Meaning $\lambda \cup \lambda = \{\mu_1, \mu_2, \dots, \mu_{l(\lambda \cup \lambda)}\}$. Thus $\mu_{2j-1} = \lambda_j$ and $\mu_{2j} = \lambda_j$.

The Pochhammer symbol on the right can be decomposed as follows

$$\begin{aligned}
[2x]_{\lambda \cup \lambda}^{(1)} &= \prod_j^{l(\lambda \cup \lambda)} \frac{\Gamma(2x - j + 1 + \mu_j)}{\Gamma(2x - j + 1)} \\
&= \prod_j^{l(\lambda)} \frac{\Gamma(2x - (2j - 1) + 1 + \lambda_j)}{\Gamma(2x - (2j - 1) + 1)} \prod_j^{l(\lambda)} \frac{\Gamma(2x - 2j + 1 + \lambda_j)}{\Gamma(2x - 2j + 1)} \\
&= \prod_j^{l(\lambda)} \frac{\Gamma(2x - 2(j - 1) + \lambda_j)}{\Gamma(2x - 2(j - 1))} \prod_j^{l(\lambda)} \frac{\Gamma(2x - 1 - 2(j - 1) + \lambda_j)}{\Gamma(2x - 1 - 2(j - 1))} \\
[2x]_{\lambda \cup \lambda}^{(1)} &= [2x]_{\lambda}^{(1/2)} [2x - 1]_{\lambda}^{(1/2)} \tag{B.55}
\end{aligned}$$

We have not assumed x to be an integer here so this identity is valid for x real. If we use this in the identity, Eq. (B.54) above, we gather

$$\frac{(1/2)^{2|\lambda|}}{d'_{\lambda}(1/2)h_{\lambda}(1/2)} = \frac{1}{h_{\lambda \cup \lambda}(1)}$$

This no longer depends on n and so it is a combinatorial relation. Thus we have by multiplying the right by $[2x]_{\lambda \cup \lambda}^{(1)}$ and the left by $[2x]_{\lambda}^{(1/2)} [2x - 1]_{\lambda}^{(1/2)}$

$$\begin{aligned}
\frac{[2x]_{\lambda \cup \lambda}^{(1)}}{h_{\lambda \cup \lambda}(1)} &= \frac{(1/2)^{2|\lambda|} [2x]_{\lambda}^{(1/2)} [2x - 1]_{\lambda}^{(1/2)}}{d'_{\lambda}(1/2)h_{\lambda}(1/2)} \\
&= \frac{e'_{\lambda}(1/2, x)b_{\lambda}(1/2, x)}{d'_{\lambda}(1/2)h_{\lambda}(1/2)} \tag{B.56}
\end{aligned}$$

B.2 Jack Polynomials

The Jack polynomials, denoted by $P_{\lambda}^{(\alpha)}(x_1, \dots, x_n)$, are multi-variable polynomials which are symmetric under the permutation of the variables and they form a basis for expanding other symmetric functions. α is a real index and in our case will be related to some kind of symmetry but we can also view them as a different set of symmetric polynomials that is orthogonal with respect to a different scalar product. When $\alpha = 1$ the Jack polynomial is equal to the Schur polynomial, $P_{\lambda}^{(\alpha)}(x_1, \dots, x_n) = S_{\lambda}(x_1, \dots, x_n)$. The variables of the Jack polynomials can also be seen as the eigenvalues of a matrix, which is our case. One has then the notation

$$P_{\lambda}^{(\alpha)}(X) = P_{\lambda}^{(\alpha)}(x_1, \dots, x_n)$$

with x_j the eigenvalues of the matrix X . For our purposes we are only interested in the Jack polynomials with $\alpha = 2, 1, \frac{1}{2}$ which corresponds in the random matrix perspective to $\beta = 1, 2, 4$ respectively ($\alpha = \frac{2}{\beta}$). The Jack polynomials evaluated at the identity matrix is known and given in terms of the Pochhammer symbol as

$$P_{\lambda}^{(\alpha)}(\mathbb{I}_n) = \frac{b_{\lambda}(\alpha, n)}{h_{\lambda}(\alpha)} = (\alpha)^{|\lambda|} \frac{[\frac{n}{\alpha}]_{\lambda}^{(\alpha)}}{h_{\lambda}(\alpha)} \tag{B.57}$$

In addition there exist relations between the different Jack polynomials evaluated at the identity and the Schur polynomials. Let us define the following partitions constructed from a partition λ .

$$\begin{aligned}\lambda &= (\lambda_1, \lambda_2, \dots) \\ 2\lambda &= (2\lambda_1, 2\lambda_2, \dots) \\ \lambda \cup \lambda &= (\lambda_1, \lambda_1, \lambda_2, \lambda_2, \dots)\end{aligned}$$

Then the following identities hold

$$\frac{e'_\lambda(\alpha, n)}{d'_\lambda(\alpha)} P_\lambda^{(\alpha)}(\mathbb{I}_n) = \begin{cases} S_\lambda(\mathbb{I}_n) & \alpha = 1 \\ S_{2\lambda}(\mathbb{I}_n) & \alpha = 2 \\ S_{\lambda \cup \lambda}(\mathbb{I}_{2n}) & \alpha = \frac{1}{2} \end{cases} \quad (\text{B.58})$$

We are mainly interested in the expansion of the determinant raised to some power. In terms Schur polynomials it is given as follows

$$\det[1 - X]^{-a} = \sum_{\lambda} S_\lambda(\mathbb{I}_a) S_\lambda(X) \quad (\text{B.59})$$

$$= \sum_{\lambda} \frac{[a]_\lambda^{(1)}}{d'_\lambda(1)} S_\lambda(X) \quad (\text{B.60})$$

In this expansion the coefficients in front of the Jack polynomials are given themselves in terms of Jack polynomials evaluated at the identity. The Zonal spherical functions defined by Macdonald [11], $\Omega_\lambda^{(\alpha)}(x)$, are defined by the following integration property

$$\int_{O(n)} dU \Omega_\lambda^{(2)}(AUB) = \Omega_\lambda^{(2)}(A) \Omega_\lambda^{(2)}(B) \quad (\text{B.61})$$

$$\int_{Sp(n)} dU \Omega_\lambda^{(1/2)}(AUB) = \Omega_\lambda^{(1/2)}(A) \Omega_\lambda^{(1/2)}(B) \quad (\text{B.62})$$

and are give in terms of Jack polynomials as

$$\Omega_\lambda^{(\alpha)}(x_n) = \frac{P_\lambda^{(\alpha)}(x_n x_n^\dagger)}{P_\lambda^{(\alpha)}(\mathbb{I}_n)} \quad (\text{B.63})$$

and can be related to the Schur functions via an integration over a group that depends on the symmetry index β .

$$\Omega_\lambda^{(2)}(x_n) = \int_{O(n)} dk S_{2\lambda}(kx) \quad (\text{B.64})$$

$$\Omega_\lambda^{(1/2)}(x) = \int_{Sp(n)} dk S_{\lambda \cup \lambda}(kx) \quad (\text{B.65})$$

A careful use of the expansion and of the integration theorems (B.61) , (B.62) , (B.64) and (B.65), is what ultimately will allow us to compute the j.p.d.f. The Hypergeometric Function of two Matrix Argument, HFMA₂, is given as follows

$${}_2\mathcal{F}_1(a, b; c | X, Y) = \sum_{\lambda} \frac{\alpha^{|\lambda|}}{d'_\lambda(\alpha)} \frac{[a]_\lambda^{(\alpha)} [b]_\lambda^{(\alpha)}}{[c]_\lambda^{(\alpha)}} \frac{P_\lambda^{(\alpha)}(X) P_\lambda^{(\alpha)}(Y)}{P_\lambda^{(\alpha)}(\mathbb{I})}. \quad (\text{B.66})$$

The Hypergeometric Function of one Matrix Argument, HFMA₁, is given as follows

$${}_2F_1(a, b; c|X) = \sum_{\lambda} \frac{\alpha^{|\lambda|}}{d'_{\lambda}(\alpha)} \frac{[a]_{\lambda}^{(\alpha)} [b]_{\lambda}^{(\alpha)}}{[c]_{\lambda}^{(\alpha)}} P_{\lambda}^{(\alpha)}(X), \quad (\text{B.67})$$

and ${}_2F_1(a, b; c|X) = {}_2\mathcal{F}_1(a, b; c|X, \mathbb{I})$.

C Average of characteristic polynomials

We wish to compute the average of characteristic polynomials of Jacobi Ensembles.

$$\prod_{j=1}^p \int_{-1}^1 d\lambda_j |\Delta(\lambda_j)|^{\beta} (1 - \lambda_j)^a (1 + \lambda_j)^b \prod_{k,j}^{n,p} (v_k - \lambda_j) \quad (\text{C.68})$$

We follow here [8] where it was found that these averages of characteristic polynomials can be written down as a pfaffian. We note that in [9] another pfaffian representation was derived. We introduce the skew-orthogonal polynomials $q_l(x)$ which satisfy the following orthogonality relations

$$\begin{aligned} \langle q_{2l}, q_{2p} \rangle &= 0 \\ \langle q_{2l+1}, q_{2p+1} \rangle &= 0 \\ \langle q_{2l}, q_{2p+1} \rangle &= r_l \delta_{lp} \end{aligned}$$

with the scalar product

$$\langle f, g \rangle = \int_{-1}^1 dv du f(v) f(u) w(v) w(u) \text{sign}(x - y) \quad (\text{C.69})$$

$$w(u) = (1 + u)^a (1 - u)^b \quad (\text{C.70})$$

There are four possible cases depending if p and n are even or odd. In our case p will be even and n arbitrary. When p is even and n even we have

$$\left\langle \prod_{k=1}^n \det[v_k - Y] \right\rangle = c_{n,p} \frac{\text{Pf}_{j,k \leq n+2} [f_{jk}]}{\Delta(v_k)} \quad (\text{C.71})$$

with f_{jk} a $n \times n$ anti symmetric matrix with the following entries

$$f_{j,k} = F(v_j, v_k) \quad (\text{C.72})$$

with

$$F(v, u) = \sum_{l=1}^{\frac{p+n}{2}} \frac{1}{2r_{l-1}} (q_{2l-2}(u) q_{2l-1}(v) - q_{2l-2}(v) q_{2l-1}(u)) \quad (\text{C.73})$$

When p is even and n is odd we have

$$\left\langle \prod_{k=1}^n \det[v_k - Y] \right\rangle = c_{n,p} \frac{\text{Pf}_{j,k \leq n+1} [f_{jk}]}{\Delta(v_k)} \quad (\text{C.74})$$

with f_{jk} a $n+1 \times n+1$ anti symmetric matrix with the following entries

$$\begin{aligned} f_{1,1} &= 0 \\ f_{j,k} &= F(v_{j-1}, v_{k-1}) \quad \text{for } j, k = 2, \dots, n+1 \\ f_{1,j+1} &= q_{n+p-1}(v_{j-1}) \quad \text{for } j = 1, \dots, n \end{aligned} \tag{C.75}$$

with

$$F(v, u) = \sum_{l=1}^{\frac{p+n-1}{2}} \frac{1}{2r_{l-1}} (q_{2l-2}(u)q_{2l-1}(v) - q_{2l-2}(v)q_{2l-1}(u)) \tag{C.76}$$

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